Problem sheet 1 –Linear ODEs and Stokes lines

1. Airy's equation is given by
$$\epsilon^2 \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - zy = 0, \tag{1}$$
which has the solution $y(z;\epsilon)$. We will solve this under the limit of $\epsilon \to 0$.

   (a) By using the Louville-Green ansatz
$$y(z;\epsilon) \sim \mathrm{e}^{-\chi_i(z)/\epsilon} \sum_{n=0}^{\infty} \epsilon^n y_n^{(i)}(z), \tag{2}$$
   obtain an equation governing $\chi_i'(z)$, as well as a set of equations governing $y_0(z)$, $y_1(z)$, and in general $y_n(z)$.
   You will need to differentiate the Louvlle-Green ansatz twice, substitute into Airys equation, and set each coefficient of ϵ^n to zero.

   (b) Solve the differential equation for $\chi(z)$ up to a constant of integration. Explain why there are only two solutions: $\chi_1(z) = 2z^{3/2}/3 + \Lambda_1$ and $\chi_2(z) = -2z^{3/2}/3 + \Lambda_2$.

   (c) Solve the equation governing $y_0(z)$. For what values of $z \in \mathbb{C}$ is the solution singular?

   (d) State the boundary conditions satisfied by $\chi_1(z)$ and $\chi_2(z)$ at $z = 0$. Explain how this condition was obtained.

   (e) Plot the $l_{1>2}$ and $l_{2>1}$ Stokes lines for this problem in $z \in \mathbb{C}$ (put the branch cut along the negative imaginary axis). In enforcing the boundary condition $\sigma_1 = a$ and $\sigma_2 = b$ along the positive real axis, obtain the asymptotic solution along the negative real axis by accounting for each Stokes line crossed from Arg$[z] = 0$ to Arg$[z] = \pi$.

2. Consider an asymptotic solution with three singulants: $\chi_1(z)$, $\chi_2(z)$, and $\chi_3(z)$.

   (a) Show that if the two Stokes lines $l_{2>1}$ and $l_{3>2}$ cross one another at a certain point in $z \in \mathbb{C}$, then a third Stokes line $l_{3>1}$ must pass through the same point (a *Stokes crossing point*).

   (b) Begin with specified values for the transseries paramters $\sigma_1 = a$, $\sigma_2 = b$, and $\sigma_3 = c$ set by boundary conditions in a certain region of $z \in \mathbb{C}$. Rotate around the Stokes crossing point, accounting for each automorphism that occurs. You should obtain a contradiction. *This contradiction occurs because the Stokes constants $\tau_{i,j}$ also exhibit a higher-order Stokes phenomenon, and do not take the same value for all $z \in \mathbb{C}$.*

3. Consider the remainder to a truncated divergent expansion,
$$R_N = \sum_{n=N+1}^{\infty} \epsilon^n y_n(z). \tag{3}$$

   (a) In taking
$$y_n(z) \sim A(z) \frac{\Gamma(n+\alpha)}{\chi(z)^{n+\alpha}}, \tag{4}$$

use the integral definition for the gamma function obtain a principal-valued integral for R_N.

Hint: Swap the order of summation and integration, and then resum the geometric series that emerges.

(b) For what value of N is the remainder minimal? Show that this is the case by expanding the integrand of the principal-valued integral above near the singular point.