Problem sheet 2 - Nonlinear differential equations

1. Consider the ordinary differential equation

$$\epsilon^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \epsilon u^2 + u = \mathrm{sech}(x), \tag{1}$$

for the solution $y(x;\epsilon)$. The real-valued domain is given by $-\infty < x < \infty$. When $\epsilon = 0$, there is an exact solitary wave solution of the form $u = \mathrm{sech}(x)$. Numerical solutions obtained with small values of ϵ contain highly oscillatory ripples in the solution, which is shown in figure 1.

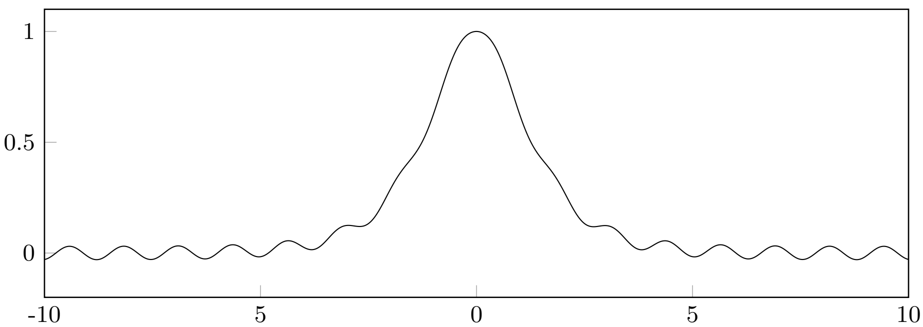

Figure 1: A symmetric numerical solution of equation (1) with $\epsilon = 0.2$ is shown.

The aim of this problem sheet is to derive this oscillatory behaviour by solving for the Stokes phenomenon that occurs in the complex plane. We will now consider the domain to be complex-valued, i.e. $x \mapsto z$, where $z \in \mathbb{C}$.

(a) By considering an asymptotic solution of the form

$$y(z;\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n y_n(z),$$

show that the equations governing $y_0(z)$, $y_1(z)$, and $y_n(z)$ for $n \geq 2$ are given by

$$\begin{aligned} O(1): &\qquad y_0(z) = \mathrm{sech}(z), \\ O(\epsilon): &\qquad y_1(z) = [y_0(z)]^2, \\ O(\epsilon^n): &\qquad y_n(z) = -y''_{n-2}(z) + \sum_{p=0}^{n-1} y_p(z) y_{n-p-1}(z). \end{aligned}$$

(b) The asymptotic solution will diverge on account of singularities in the leading-order solution $y_0(z)$. By writing $\mathrm{sech}(z) = 1/\mathrm{cosh}(z)$, identify the points $z^* \in \mathbb{C}$ at which $\mathrm{cosh}(z^*) = 0$.

(c) What is the strength of the singularity in $y_0(z)$ at $z = z^*$? I.e., identify the value of the constant a in

$$y_0(z) \sim \frac{\lambda}{(z - z^*)^a} \quad \text{as} z \to z^*.$$

Obtain the singular behaviour for $y_n(z)$.
You might want to consider $y_1(z)$, $y_2(z)$, and then spot the general pattern for the strength of this singularity.

(d) We now derive the divergence of $y_n(z)$ as $n \to \infty$. By assuming that

$$y_n(z) \sim A(z) \frac{\Gamma(n+\alpha)}{\chi(z)^{n+\alpha}} \quad \text{as } n \to \infty,$$

Show that as $n \to \infty$, the two dominant orders in the $O(\epsilon^n)$ equation are given by

$$y_n(z) = -y''_{n-2}(z) + 2y_0(z)y_{n-1}(z).$$

By substituting the divergent ansatz into this equation and examining the coefficients of different orders in n, obtain the equations $[\chi'(z)]^2 = -1$ and $A'(z)\chi'(z) + y_0(z)A(z) = 0$.

(e) Show that the solutions for $\chi(z)$ are given by

$$\chi(z) = \pm\mathrm{i}(z - z^*),$$

where z^* is a singular point of $y_0(z)$. We now consider only the two singular points closest to the real axis, $z^* = \pm\mathrm{i}\pi/2$. In defining

$$\chi_2 = +\mathrm{i}(z - \mathrm{i}\pi/2), \quad \chi_3 = -\mathrm{i}(z - \mathrm{i}\pi/2), \quad \chi_4 = +\mathrm{i}(z + \mathrm{i}\pi/2), \quad \chi_5 = -\mathrm{i}(z + \mathrm{i}\pi/2),$$

sketch the Stokes lines $l_{1>2}, l_{1>3}, l_{1>4}, l_{1>5}$ for this problem. Which two Stokes lines intersect with the real axis? Note that $\chi_1(z) = 0$ here.

(f) Now consider the exponentially subdominant component of the asymptotic expansion switched on by the Stokes phenomenon. In splitting the two relevent singulant functions (χ_i) into real and imaginary parts, show that $\mathrm{e}^{-\chi(z)/\epsilon}$ is exponentially small, and oscillatory under the limit of $\epsilon \to 0$.

In writing the asymptotic solution as

$$y(z;\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n y_n(z) + \frac{\sigma_i}{\epsilon^{\alpha_i}} A(z)\mathrm{e}^{-\chi_i(z)/\epsilon},$$

explain how different boundary conditions on σ_i (imposed at $x = -\infty$) produce each of the three numerical solutions shown below.