Random walks, electrical networks and uniform spanning trees

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1 Transience and recurrence

Definition 1.1. A sequence of random variables $(X_n)_{n\geq 0}$ taking values in a space E is called a Markov chain if for all $x_0, \ldots, x_n \in E$ such that $\mathbb{P}(X_0 = x_0, \ldots, X_{n-1} = x_{n-1}) > 0$ we have

$$\mathbb{P}(X_n = x_n \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}) = \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}) .$$

In other words, the future of the process is independent of the past given the present.

For an event A we write $\mathbb{P}_i(A)$ to denote $\mathbb{P}(A \mid X_0 = i)$.

A Markov chain is defined by its transition matrix P given by

$$P(i,j) = \mathbb{P}(X_1 = j \mid X_0 = i) \quad \forall\, i,j \in E.$$

Exercise 1.2. *Check that*

$$\mathbb{P}_i(X_n = j) = P^n(i,j),$$

where P^n is the n-th power of the matrix P.

We will also write $p_{i,j}(n)$ or $p_n(i,j)$ for $P^n(i,j)$.

Lemma 1.3. *Suppose $X_0 = i$. The number of visits to i, i.e. $V_i = \sum_{n=0}^{\infty} \mathbf{1}(X_n = i)$, is a geometric random variable with parameter $\mathbb{P}_i(T_i < \infty)$, where $T_i = \inf\{n \geq 1 : X_n = i\}$.*

Proof. Define the successive visits to i via $T_i^{(1)} = T_i$ and inductively for $j \geq 2$

$$T_i^{(j)} = \inf\{n > T_i^{(j-1)} : X_n = i\}.$$

Then by the definition of a Markov chain we have

$$\mathbb{P}\left(T_i^{(j)} < \infty \;\middle|\; T_i^{(j-1)} < \infty\right) = \mathbb{P}_i(T_i < \infty) .$$

This shows that the number of visits has the geometric distribution with parameter $\mathbb{P}_i(T_i < \infty)$. □

Definition 1.4. A Markov chain is called irreducible if for all $x, y \in E$ there exists $n \geq 0$ such that $P^n(x,y) > 0$.

An irreducible Markov chain is called recurrent if for all i we have $\mathbb{P}_i(T_i < \infty) = 1$. Otherwise, it is called transient.

Exercise 1.5. *Suppose that X is an irreducible Markov chain. Suppose that there exists i such that $\mathbb{P}_i(T_i < \infty) = 1$. Show that for all j we also have $\mathbb{P}_j(T_j < \infty) = 1$.*

From now on, all the Markov chains under consideration will be irreducible unless otherwise specified.

Theorem 1.6. *Let X be a Markov chain and $i \in E$. Then X is recurrent if and only if*

$$\sum_{n=0}^{\infty} p_n(i,i) = \infty.$$

Proof. Let V_i be the total number of visits to i. Then $\mathbb{E}_i[V_i] = \sum_{n=0}^{\infty} p_n(i,i)$. The statement follows from Lemma 1.3. $\square$

Let $G = (V, E)$ be a connected graph, which may be infinite or finite. A simple random walk on G is a Markov chain evolving on the vertices V with transition matrix given by

$$P(i,j) = \frac{1}{\deg(i)},$$

for i and j neighbours, i.e. joined by an edge.

Theorem 1.7 (Polya). *Let $G = \mathbb{Z}^d$. Then simple random walk on G is recurrent if and only if $d \le 2$.*

Proof. We start with $d = 1$. Then we have

$$p_{2n}(0,0) = \binom{2n}{n} \cdot \frac{1}{2^{2n}} \sim \frac{c}{\sqrt{n}},$$

for a positive constant c using Stirling's formula. Invoking Theorem 1.6 gives the recurrence for $d = 1$.

For $d = 2$, we project the walk on the two diagonals and resize the lattice by dividing by $\sqrt{2}$. Then each component becomes an independent simple random walk on each diagonal and the result follows using what we showed for $d = 1$, i.e.

$$\sum_{n=0}^{\infty} p_n(0,0) \sim \sum_{n=1}^{\infty} \frac{c}{n} = \infty.$$

For $d = 3$ we have

$$p_{2n}(0,0) = \sum_{\substack{i,j,k \ge 0 \\ i+j+k=n}} \frac{(2n)!}{(i!j!k!)^2} \cdot \left(\frac{1}{6}\right)^{2n} = \binom{2n}{n} \cdot \left(\frac{1}{2}\right)^{2n} \sum_{\substack{i,j,k \ge 0 \\ i+j+k=n}} \binom{n}{i\ j\ k}^2 \cdot \left(\frac{1}{3}\right)^{2n}.$$

Now notice that by considering all possible ways of placing n balls into three boxes we get

$$\sum_{\substack{i,j,k \ge 0 \\ i+j+k=n}} \binom{n}{i\ j\ k} \cdot \left(\frac{1}{3}\right)^{n} = 1.$$

By a simple counting argument we obtain when $n = 3m$ for all i, j, k with $i + j + k = n$

$$\binom{n}{i\ j\ k} \le \binom{n}{m\ m\ m}.$$

Using that $p_{6m}(0,0) \ge (1/6)^2 p_{6m-2}(0,0)$ and $p_{6m}(0,0) \ge (1/6)^4 p_{6m-4}(0,0)$ for all m we deduce using again Stirling's formula

$$\sum_n p_{2n}(0,0) \le \sum_n \frac{c}{n^{3/2}} < \infty,$$

which shows that the walk is transient. $\square$

Exercise 1.8. *Show that simple random walk on $\mathbb{Z}^d$ for all $d \geq 4$ is transient.*

2 Invariant distribution

Let E be a countable (infinite or finite) state space and let π be a probability distribution on E. We call π an invariant distribution if $\pi P = P$. This means that if $X_0 \sim \pi$, then $X_n \sim \pi$ for all n.

A Markov chain X is called reversible if for all N when $X_0 \sim \pi$, then $(X_0, \ldots, X_N)$ has the same distribution as $(X_N, \ldots, X_0)$.

Exercise 2.1. *Show that a chain with matrix P and invariant distribution π is reversible if and only if for all x, y we have*

$$\pi(x)P(x, y) = \pi(y)P(y, x).$$

Exercise 2.2. *Show that if X is reversible, then for all a, b, c we have*

$$\mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_c] + \mathbb{E}_c[\tau_a] = \mathbb{E}_a[\tau_c] + \mathbb{E}_c[\tau_b] + \mathbb{E}_b[\tau_a]$$

Exercise 2.3. *Let X be a simple symmetric random walk on $\mathbb{Z}_n$. Find its invariant distribution. Is the chain reversible?*

Let X be a biased random walk on $\mathbb{Z}_n$ with transition probabilities $P(i, (i+1) \bmod n) = 2/3$ and $P(i, (i-1) \bmod n) = 1/3$. Find its invariant distribution. Is the chain reversible?

Consider next the biased random walk on $\{0, \ldots, n\}$, i.e. with transition probabilities $P(i, i+1) = 2/3 = 1 - P(i, i-1)$. Is this chain reversible?

3 Random walks on graphs

In this section we are following closely [2, Chapter 9].

Let $G = (V, E)$ be a finite connected graph with set of vertices V and set of edges E. We endow it with non-negative numbers $(c(e))_{e \in E}$ that we call conductances. We write $c(x, y) = c(\{x, y\})$ and clearly $c(x, y) = c(y, x)$. The reciprocal $r(e) = 1/c(e)$ is called the resistance of the edge e.

We now consider the Markov chain on the nodes of G with transition matrix

$$P(x, y) = \frac{c(x, y)}{c(x)},$$

where $c(x) = \sum_{y: y \sim x} c(x, y)$. This process is called the weighted random walk on G with edge weights $(c(e))_e$.

This process is reversible with respect to $\pi(x) = c(x)/c_G$, where $c_G = \sum_{x \in V} c(x)$, since

$$\pi(x)P(x, y) = \frac{c(x)}{c_G} \cdot \frac{c(x, y)}{c(x)} = \frac{c(x, y)}{c_G} = \pi(y)P(y, x)$$

and π is stationary for P, i.e. $\pi P = \pi$.

When $c(e) = 1$ for all edges e, we call the Markov chain with transition matrix P a simple random walk on G. In this case

$$P(x,y) = \begin{cases} \frac{1}{d(x)} & \text{if } y \sim x, \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to show that every reversible Markov chain is a weighted random walk on a graph. Indeed, suppose that P is a transition matrix which is reversible with respect to the stationary distribution π. Then we declare $\{x, y\}$ an edge if $P(x, y) > 0$. Reversibility implies that $P(y, x) > 0$ if $P(x, y) > 0$, and hence this is well-defined. We define conductances on the edges by setting $c(x, y) = \pi(x)P(x, y)$. Again by reversibility this is symmetric and with this choice of weights we get $\pi(x) = c(x)$. The study of reversible Markov chains is thus equivalent to the study of random walks on weighted graphs.

Let now P be a transition matrix which is irreducible with state space Ω. (We do not assume that P is reversible.)

A function $h : \Omega \to \mathbb{R}$ is called harmonic for P at the vertex x if

$$h(x) = \sum_{y \in \Omega} P(x,y)h(y).$$

For a subset $B \subseteq \Omega$ we define the hitting time of B by

$$\tau_B = \min\{t \geq 0 : X_t \in B\}.$$

Proposition 3.1. *Let X be an irreducible Markov chain with transition matrix P and let $B \subseteq \Omega$. Let $f : B \to \mathbb{R}$ be a function defined on B. Then the function $h(x) = \mathbb{E}_x[f(X_{\tau_B})]$ is the unique extension $h : \Omega \to \mathbb{R}$ of f such that $h(x) = f(x)$ for all $x \in B$ and h is harmonic for P at all $x \in \Omega \setminus B$.*

Proof. Clearly $h(x) = f(x)$ for all $x \in B$. By conditioning on the first step of the Markov chain we get for $x \notin B$

$$h(x) = \sum_{y \in \Omega} P(x,y)\mathbb{E}_x[f(X_{\tau_B}) \mid X_1 = y] = \sum_y P(x,y)\mathbb{E}_y[f(X_{\tau_B})] = \sum_y P(x,y)h(y),$$

where for the second equality we used the Markov property.

We now turn to show uniqueness. Let h' be another function satisfying the same conditions as h. Then the function $g = h - h'$ is harmonic on $\Omega \setminus B$ and $g = 0$ on B. We first show that $g \leq 0$. Consider the set

$$A = \left\{ x : g(x) = \max_{y \in \Omega} g(y) \right\}.$$

If $x \in A$ and $x \in B$, then we are done. Suppose next that $x \notin B$ and let y be such that $P(x, y) > 0$. If $g(y) < g(x)$, then harmonicity of g on $\Omega \setminus B$ implies that

$$g(x) = \sum_{z \in \Omega} g(z)P(x,z) = g(y)P(x,y) + \sum_{z \neq y} P(x,z)g(z) < \max_{y \in \Omega} g(y),$$

which is clearly a contradiction. Hence it follows that $g(y) = \max_z g(z)$, which means that $y \in A$.

By irreducibility we continue in the same way and we finally get that $g(x) = 0$, since we eventually get to the boundary B. Hence this proves that $\max g = 0$. Similarly, we can prove that $\min g = 0$. So we deduce that $g = 0$. $\square$

3.1 Electrical networks

The goal of this section is to explain the connection between random walks and electrical networks.

Again here $G = (V, E)$ will be a finite connected graph with conductances $(c(e))_e$ on the edges. We distinguish two vertices a and b that will be the source and the sink respectively.

Definition 3.2. A flow θ on G is a function defined on oriented edges $\vec{e} = (x, y)$ of E satisfying

$$\theta(x, y) = -\theta(y, x).$$

The **divergence** of the flow θ is defined to be $\operatorname{div}\theta(x) = \sum_{y\sim x} \theta(x, y)$.

We note that by the antisymmetric property of θ we get

$$\sum_x \operatorname{div}\theta(x) = \sum_{(x,y)\in E} (\theta(x, y) + \theta(y, x)) = 0.$$

Definition 3.3. A **flow** θ **from** a **to** b is a flow such that

- $\operatorname{div}\theta(x) = 0 \quad \forall\, x \notin \{a, b\}$ ("flow in equals flow out" – Kirchoff's node law)
- $\operatorname{div}\theta(a) \geq 0$.

The **strength** of a flow θ from a to b is defined to be $\|\theta\| := \operatorname{div}\theta(a)$. A **unit flow** is a flow with $\|\theta\| = 1$.

We note that $\operatorname{div}\theta(b) = -\operatorname{div}\theta(a)$.

A **voltage** W is a harmonic function on $V \setminus \{a, b\}$. By Proposition 3.1 a voltage always exists and is uniquely determined by its boundary values $W(a)$ and $W(b)$.

Given a voltage W on G we define the **current flow** I on oriented edges via

$$I(x, y) = \frac{W(x) - W(y)}{r(x, y)} = c(x, y)(W(x) - W(y)).$$

Exercise: Check it is a flow and then prove that the unit current flow is unique.

By the definition we immediately see that the current flow satisfies Ohm's law:

$$r(x, y)I(x, y) = W(x) - W(y).$$

The current flow also satisfies the **cycle law**: if the oriented edges $\vec{e}_1, \ldots, \vec{e}_n$ form an oriented cycle, then

$$\sum_{i=1}^{n} r(\vec{e}_i)I(\vec{e}_i) = 0.$$

Proposition 3.4. *Let I be a current and θ a flow from a to z satisfying the cycle law for any cycle. If $\|\theta\| = \|I\|$, then $\theta = I$.*

Proof. Consider the function $f = \theta - I$. Then since θ and I are flows with the same strength, it follows that f satisfies Kirchoff's node law at all nodes and the cycle law. Suppose that $\theta \neq I$. Then without loss of generality, there must exist an edge $\vec{e}_1$ such that $f(\vec{e}_1) > 0$. Since $\sum_{y \sim x} f(x, y) = 0$, we get that there must exist an edge $\vec{e}_2$ to which $\vec{e}_1$ leads such that $f(\vec{e}_2) > 0$. Continuing in this way we get a sequence of oriented edges with positive value of f. Since the graph is finite, at some point this sequence of edges should revisit a point. This then violates the cycle law. Hence $\theta = I$. □

3.2 Effective resistance

Let W_0 be the voltage with $W_0(a) = 1$ and $W_0(z) = 0$. By the uniqueness of harmonic functions, we obtain that any other voltage W is given by

$$W(x) = (W(a) - W(z))W_0(x) + W(z).$$

We call I_0 the current flow associated with W_0. Note that by the definition of the current flow, its strength is given by

$$\|I\| = \sum_{x \sim a} \frac{W(a) - W(x)}{r(a, x)} = (W(a) - W(z)) \|I_0\|.$$

We thus see that the ratio

$$\frac{W(a) - W(z)}{\|I\|}$$

is independent of W. We define this to be the effective resistance

$$R_{\text{eff}}(a, z) := \frac{W(a) - W(z)}{\|I\|}$$

and the reciprocal is called the effective conductance, $C_{\text{eff}}(a, z)$.

Why is it called effective resistance? Suppose that we wanted to replace the whole network by single edge joining a and z with resistance $R_{\text{eff}}(a, z)$. Then if we apply the same voltage at a and z in both networks, then the same amount of current would flow through.

We are now ready to state the connection between random walks and electrical networks.

We define $\tau_x^+ = \min\{t \geq 1 : X_t = x\}$.

Proposition 3.5. *Let X be a reversible chain on a finite state space. For any $a, z \in \Omega$ we have*

$$\mathbb{P}_a(\tau_z < \tau_a^+) = \frac{1}{c(a) R_{\text{eff}}(a, z)}.$$

Proof. The function $f(x) = \mathbb{P}_x(\tau_z < \tau_a)$ is a harmonic function on $\Omega \setminus \{a, z\}$ and $f(a) = 0$ and $f(z) = 1$. So from Proposition 3.1 we get that f has to be equal to the function

$$h(x) = \frac{W(a) - W(x)}{W(a) - W(z)},$$

where W is a voltage, since they are both harmonic with the same boundary values. Therefore, we obtain

$$\mathbb{P}_a\big(\tau_z < \tau_a^+\big) = \sum_x P(a,x)\mathbb{P}_x(\tau_z < \tau_a) = \sum_{x\sim a} \frac{c(a,x)}{c(a)} \cdot \frac{W(a)-W(x)}{W(a)-W(z)}.$$

By the definition of the current flow, the above sum is equal to

$$\frac{\sum_{x\sim a} I(a,x)}{c(a)(W(a)-W(z))} = \frac{1}{c(a)R_{\text{eff}}(a,z)}$$

and this proves the proposition. □

Definition 3.6. The **Green's function** for a random walk stopped at a stopping time τ is defined to be

$$G_\tau(a,x) := \mathbb{E}_a\left[\sum_{t=0}^{\infty} \mathbf{1}(X_t = x, \tau > t)\right].$$

Lemma 3.7. *Let X be a reversible Markov chain. Then for all a, z we have*

$$G_{\tau_z}(a,a) = c(a)R_{\text{eff}}(a,z).$$

Proof. The number of visits to a before the first hitting time of z has a geometric distribution with parameter $\mathbb{P}_a(\tau_z < \tau_a^+)$. The statement follows from Proposition 3.5. □

There are some ways of simplifying a network without changing quantities of interest.

Conductances in parallel add: let e_1 and e_2 be edges sharing the same endvertices. Then we can replace both edges by a single edge of conductance equal to the sum of the conductances. Then the same current will flow through and the same voltage difference will be applied. To see it, check Kirchoff's and Ohm's laws with $I(\vec{e}) = I(\vec{e}_1) + I(\vec{e}_2)$.

Resistances in series add: if $v \in V \setminus \{a, z\}$ is a node of degree 2 with neighbours v_1 and v_2, we could replace the edges (v, v_1) and (v, v_2) by a single edge (v_1, v_2) of resistance $r(v_1, v_2) = r(v, v_1) + r(v, v_2)$. To see it, check Kirchoff's and Ohm's laws with $I(v_1, v_2) = I(v_1, v) = I(v, v_2)$ and the same as before everywhere else.

Gluing: If two vertices have the same voltage, we can glue them to a single vertex, while keeping all existing edges. Since current never flows between vertices with the same voltage, potentials and currents remain unchanged.

Example 3.8. Let a and b be two vertices on a finite connected tree T. Then the effective resistance $R_{\text{eff}}(a,b)$ is equal to the distance on the tree between a and b.

Definition 3.9. Let θ be a flow on a finite connected graph G. We define its energy by

$$\mathcal{E}(\theta) = \sum_e (\theta(e))^2 r(e),$$

where the sum is taken over all unoriented edges e. (Note that since θ is antisymmetric, we did not need to take direction on e.)

The following theorem gives an equivalent definition of effective resistance as the minimal energy over all flows of unit strength from a to z.

Theorem 3.10 (Thomson's principle)**.** *Let G be a finite connected graph with edge conductances $(c(e))_e$. For all a and z we have*

$$R_{\text{eff}}(a,z) = \inf\{\mathcal{E}(\theta) : \theta \textit{ is a unit flow from } a \textit{ to } z\}.$$

The unique minimiser above is the unit current flow from a to z.

Proof. We follow [1].

Let i be the unit current flow from a to z associated to the potential φ.

We start by showing that

$$R_{\text{eff}}(a,z) = \mathcal{E}(i).$$

Using that i is a flow from a to z and Ohm's law we have

$$\mathcal{E}(i) = \frac{1}{2}\sum_{\substack{u,v\\ u\sim v}} i(u,v)^2 r(u,v) = \frac{1}{2}\sum_{\substack{u,v\\ u\sim v}} i(u,v)(\varphi(u)-\varphi(v)) = \varphi(a)-\varphi(z) = R_{\text{eff}}(a,z).$$

Let j be another flow from a to z of unit strength. The goal is to show that $\mathcal{E}(j) \geq \mathcal{E}(i)$.

We define $k = j - i$. Then this is a flow of 0 strength. So we now get

$$\begin{aligned}
\mathcal{E}(j) &= \sum_e (j(e))^2 r(e) = \sum_e (i(e)+k(e))^2 r(e) \\
&= \sum_e (i(e))^2 r(e) + \sum_e (k(e))^2 r(e) + 2\sum_e k(e)i(e)r(e) \\
&= \mathcal{E}(i) + \mathcal{E}(k) + 2\sum_e k(e)i(e)r(e).
\end{aligned}$$

We now show that

$$\sum_e k(e)i(e)r(e) = 0.$$

Since i is the unit current flow associated with φ, for $e=(x,y)$ it satisfies

$$i(x,y) = \frac{\varphi(x)-\varphi(y)}{r(x,y)}.$$

Substituting this above we obtain

$$\sum_e k(e)i(e)r(e) = \frac{1}{2}\cdot\sum_x\sum_{y\sim x}(\varphi(x)-\varphi(y))k(x,y) = \frac{1}{2}\cdot\sum_x\sum_{y\sim x}\varphi(x)k(x,y) + \frac{1}{2}\cdot\sum_x\sum_{y\sim x}\varphi(x)k(x,y),$$

where for the last equality we used the antisymmetric property of k. Since k is a flow of 0 strength, we get that both these sums are equal to 0. Therefore this proves that

$$\mathcal{E}(j) \geq \mathcal{E}(i)$$

with equality if and only if $\mathcal{E}(k) = 0$ which is equivalent to $k = 0$. □

Theorem 3.11 (Rayleigh monotonicity principle)**.** *The effective resistance is a monotone increasing function as a function of the component resistances, i.e. if $(r(e))_e$ and $(r'(e))_e$ satisfy $r(e) \le r'(e)$ for all e, then*

$$R_{\text{eff}}(a, z; r) \le R_{\text{eff}}(a, z; r').$$

Proof. Let i and i' be the unit current flows associated to the resistances $r(e)$ and $r'(e)$ respectively. Then by Thomson's principle we get

$$R_{\text{eff}}(a, z; r) = \sum_e (i(e))^2 r(e) \le \sum_e (i'(e))^2 r(e),$$

where the inequality follows, since the energy is minimised by the unit current flow i. Using the assumption on the resistances we now obtain

$$\sum_e (i'(e))^2 r(e) \le \sum_e (i'(e))^2 r'(e) = R_{\text{eff}}(a, z; r')$$

and this concludes the proof. □

Corollary 3.12. *Let G be a finite connected graph. Suppose we add an edge which is not adjacent to a. Then this increases the escape probability $\mathbb{P}_a(\tau_z < \tau_a^+)$.*

Proof. Recall from Proposition 3.5 that

$$\mathbb{P}_a\big(\tau_z < \tau_a^+\big) = \frac{1}{c(a) R_{\text{eff}}(a, z)}.$$

Adding an edge means that we decrease the resistance of the edge from ∞ to a finite number. Hence from Rayleigh's monotonicity principle we get that the effective resistance will decrease. □

Corollary 3.13. *The operation of gluing vertices together cannot increase the effective resistance.*

We now present a nice technique due to Nash and Williams to obtain lower bounds on effective resistances.

Definition 3.14. We call a set of edges Π an edge–cutset separating a from z if every path from a to z uses an edge of Π.

Proposition 3.15 (Nash-Williams inequality)**.** *If (Π_k) are disjoint edge-cutsets which separate a from z, then*

$$R_{\text{eff}}(a, z) \ge \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

Proof. By Thomson's principle it suffices to prove that for any flow θ from a to z of unit strength we have

$$\sum_e (\theta(e))^2 r(e) \ge \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

Since the sets Π_k are disjoint, we get

$$\sum_e (\theta(e))^2 r(e) \geq \sum_k \sum_{e \in \Pi_k} (\theta(e))^2 r(e).$$

By the Cauchy-Schwarz inequality we now get

$$\left(\sum_{e \in \Pi_k} c(e) \right) \cdot \left(\sum_{e \in \Pi_k} r(e)(\theta(e))^2 \right) \geq \left(\sum_{e \in \Pi_k} \sqrt{c(e)} \sqrt{r(e)} |\theta(e)| \right)^2 = \left(\sum_{e \in \Pi_k} |\theta(e)| \right)^2.$$

But since the sets Π_k are cutsets separating a from z and θ has unit strength, this last sum is at least 1. Rearranging completes the proof. □

Proposition 3.16. *Let $a = (1,1)$ and $z = (n,n)$ be the opposite corners of the box $B_n = [1,n]^2 \cap \mathbb{Z}^2$. Then*

$$R_{\text{eff}}(a,z) \geq \frac{\log(n-1)}{2}.$$

Remark 3.17. The effective resistance between a and z is also upper bounded by $\log n$. To prove it one defines a flow and shows that its energy is at most $\log n$. For more details see [2, Proposition 9.5].

Proof of Proposition 3.16. Let $\Pi_k = \{(v,u) \in B_n : \|v\|_\infty = k \text{ and } \|u\|_\infty = k-1\}$. Then (Π_k) are disjoint edge-cutsets separating a from z. By counting the number of edges we get $|\Pi_k| = 2(k-1)$ and since $c(e) = 1$ for all edges e we get

$$R_{\text{eff}}(a,z) \geq \sum_{k=2}^{n} \frac{1}{2(k-1)} \geq \frac{\log(n-1)}{2}$$

and this completes the proof. □

Lemma 3.18. *Let X be an irreducible Markov chain on a finite state space. Let τ be a stopping time satisfying $\mathbb{E}[\tau] < \infty$ and $\mathbb{P}_a(X_\tau = a) = 1$ for some a in the state space. Then for all x we have*

$$G_\tau(a,x) = \mathbb{E}_a[\tau] \cdot \pi(x).$$

Proposition 3.19 (Commute time identity)**.** *Let X be a reversible Markov chain on a finite state space. Then for all a, b we have*

$$\mathbb{E}_a[\tau_{a,b}] = \mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a] = c(G) \cdot R_{\text{eff}}(a,b),$$

where $c(G) = 2\sum_e c(e)$.

Proof. The stopping time $\tau_{a,b}$ which is the first time the walk comes back to a after having visited b satisfies $\mathbb{P}_a(X_{\tau_{a,b}} = a) = 1$. Also note that we only visit a before time τ_b, i.e.

$$G_{\tau_{a,b}}(a,a) = G_{\tau_b}(a,a) = c(a) R_{\text{eff}}(a,b),$$

where the last equality follows from Proposition 3.5. We can now apply the previous lemma to finish the proof. □

Remark 3.20. The previous identity immediately gives us that the effective resistance satisfies the triangle inequality. Hence the effective resistance defines a metric on the graph G (the other two properties are trivially satisfied).

3.3 Transience vs recurrence

So far we have been focusing on finite state spaces. In this section we will see how we can use the electrical network point of view to determine transience and recurrence properties of graphs.

Let $G = (V, E)$ be a countable graph and let 0 be a distinguished point. Let $G_k = (V_k, E_k)$ be an exhaustion of G by finite graphs, i.e. $V_n \subseteq V_{n+1}$ and E_n contains all edges of E with endpoints in V_n for all n, $0 \in V_n$ for all n and $\cup_n V_n = V$.

For every n we construct a graph G_n^* by gluing all the points of $V \setminus V_n$ into a single point z_n. We now define

$$R_{\text{eff}}(0, \infty) = \lim_{n\to\infty} R_{\text{eff}}(0, z_n; G_n^*).$$

Check by Rayleigh's monotonicity principle that this limit does exist and is independent of the choice of exhaustion. We thus have

$$\mathbb{P}_0\big(\tau_0^+ = \infty\big) = \lim_{n\to\infty} \mathbb{P}_0\big(\tau_{z_n} < \tau_0^+\big) = \lim_{n\to\infty} \frac{1}{c(0)R_{\text{eff}}(0, z_n; G_n^*)} = \frac{1}{c(0)R_{\text{eff}}(0, \infty)}. \tag{3.1}$$

We can define a flow from 0 to ∞ on an infinite graph as an antisymmetric function on the edges with $\text{div}\theta(\text{x}) = 0$ for all $x \neq 0$.

Proposition 3.21. *Let G be a countable connected weighted graph with conductances $(c(e))_e$ and let 0 be a distinguished vertex.*

(a) *A random walk on G is recurrent if and only if $R_{\text{eff}}(0, \infty) = \infty$.*

(b) *A random walk on G is transient if and only if there exists a unit flow i from 0 to ∞ of finite energy $\mathcal{E}(i) = \sum_e (i(e))^2 r(e) < \infty$.*

Proof. (a) The first part follows directly from (3.1). If the walk is recurrent, then

$$\mathbb{P}_0\big(\tau_0^+ = \infty\big) = 0,$$

and hence $R_{\text{eff}}(0, \infty) = \infty$ and vice versa.

(b) For the second part let G_n be an exhaustion of G by finite graphs. Then by definition

$$R_{\text{eff}}(0, \infty) = \lim_{n\to\infty} R_{\text{eff}}(0, z_n).$$

Let i_n be the unit current flow from 0 to z_n on the graph G_n and let v_n be the corresponding voltage. Then by Thomson's principle we get

$$R_{\text{eff}}(0, z_n) = \mathcal{E}(i_n).$$

Suppose now that there exists a unit flow θ from 0 to ∞ of finite energy. We call θ_n the restriction of θ to the graph G_n. Then θ_n is a unit flow from a to z_n in G_n^*. Applying Thomson's principle we obtain

$$\mathcal{E}(i_n) \leq \mathcal{E}(\theta_n) \leq \mathcal{E}(\theta) < \infty.$$

Therefore we get

$$R_{\text{eff}}(0, \infty) = \lim_{n \to \infty} \mathcal{E}(i_n) < \infty,$$

which implies that the walk is transient from the first part.

Suppose now that the walk is transient. Then $R_{\text{eff}}(0, \infty) < \infty$ from the first part. We want to construct a unit flow from 0 to ∞ of finite energy. Since $R_{\text{eff}}(0, \infty) < \infty$ we get that $\lim_{n\to\infty} \mathcal{E}(i_n) < \infty$, so there exists $M > 0$ such that $\mathcal{E}(i_n) \leq M$ for all n.

We now start a random walk from 0 and call $Y_n(x)$ the number of visits to x up until it hits z_n. We also call $Y(x)$ the total number of visits to x. It is clear that $Y_n(x) \uparrow Y(x)$ as $n \to \infty$, and hence by monotone convergence we get

$$\lim_{n \to \infty} \mathbb{E}_0[Y_n(x)] \uparrow \mathbb{E}_0[Y(x)] \,.$$

Since the walk is assumed to be transient, we have $\mathbb{E}_0[Y(x)] < \infty$. It is not hard to check that the function $G_{\tau_{z_n}}(0, x)/c(x)$ is a harmonic function with value 0 at z_n, and it is equal to $v_n(x) - v_n(z_n)$. (Use reversibility and same proof as in Proposition 3.1). So we get

$$\lim_{n \to \infty} c(x)(v_n(x) - v_n(z_n)) = c(x)v(x)$$

for some function v which is finite. Therefore, we can define

$$i(x, y) = c(x, y)(v(x) - v(y)) = \lim_{n \to \infty} c(x, y)(v_n(x) - v_n(y)).$$

Since $\mathcal{E}(i_n) \leq M$ for all n using dominated convergence one can show that i is a unit flow from 0 to ∞ of finite energy (Check!). □

Corollary 3.22. *If $G \subseteq G'$ and G' is recurrent, then G is also recurrent. If G is transient, then G' is also transient.*

Corollary 3.23. *Simple random walk is recurrent on $\mathbb{Z}^2$ and transient on $\mathbb{Z}^d$ for all $d \geq 3$.*

Proof. For $d = 2$ we construct a new graph in which for each k we identify all vertices at distance k from 0. By the series/parallel law we see that

$$R_{\text{eff}}(0, \partial\Lambda_n) \geq \sum_{i=1}^{n-1} \frac{1}{8i - 4}.$$

Therefore, we get that

$$R_{\text{eff}}(0, n) \geq c \log n \to \infty \quad \text{ as } n \to \infty.$$

For $d = 3$ we are going to construct a flow of finite energy. To each directed edge of $\mathbb{Z}^3$ we attach an orthogonal unit square intersecting e at its midpoint m_e. We now define the absolute value of

$\theta(\vec{e})$ to be the area of the radial projection of this square onto the sphere of radius $1/4$ centred at the origin. We take $\theta(\vec{e})$ with positive sign if $\vec{e}$ points in the same direction as the radial vector from 0 to m_e and negative otherwise. By considering the projections of all faces of the unit cube centred at a lattice point, we can see that θ satisfies Kirchoff's node law at all vertices except for 0 (Check!). Hence θ is a flow from 0 to ∞ in $\mathbb{Z}^3$. Its energy is given by

$$\mathcal{E}(\theta) \leq \sum_n c_1 n^2 \cdot \left(\frac{c_2}{n^2}\right)^2 < \infty,$$

and hence this proves transience.

An alternative proof goes via embedding a binary tree with resistance between edges from level $n-1$ to n equal to ρ^n for a suitable $\rho > 0$. Then the effective resistance of this tree is given by

$$R_{\text{eff}}(0,\infty) = \sum_{n=1}^{\infty} \left(\frac{\rho}{2}\right)^n.$$

Taking $\rho < 2$ makes it finite.

We now want to embed this tree in $\mathbb{Z}^3$ in such a way that a vertex at distance $n-1$ and a neighbour at distance n are separated by a path of length ρ^n.

The surface of a ball of radius k in $\mathbb{R}^3$ is of order k^2, so in order to be able to accommodate this tree in $\mathbb{Z}^3$ we need

$$(\rho^n)^2 \geq 2^n,$$

which then gives $\rho > \sqrt{2}$. This then gives that the effective resistance from 0 to ∞ in $\mathbb{Z}^3$ is bounded by the effective resistance of the tree, and hence it is finite.

This idea has been used to show that random walk on the infinite component of supercritical percolation cluster is transient in dimensions $d \geq 3$ (see Grimmett, Kesten and Zhang (1993)). □

3.4 Spanning trees

Definition 3.24. Let G be a finite connected graph. A spanning tree of G is a subgraph that is a tree (no cycles) and which contains all the vertices of G.

Let G be a finite connected graph and let $\mathcal{T}$ be the set of spanning trees of G. We pick T uniformly at random from $\mathcal{T}$. We call T a uniform spanning tree (UST).

We will prove that T has the property of negative association, i.e.

Theorem 3.25. *Let $G = (V,E)$ be a finite graph. Let $f, g \in E$ with $f \neq g$. Let T be a UST. Then*

$$\mathbb{P}(f \in T \mid g \in T) \leq \mathbb{P}(f \in T).$$

In order to prove this theorem we are first going to establish a connection between spanning trees and electrical networks.

Let $\mathcal{N}(s,a,b,t)$ be the set of spanning trees of G with the property that the unique path from s to t passes along the edge (a,b) in the direction from a to b. We write $N(s,a,b,t) = |\mathcal{N}(s,a,b,t)|$.

Let N be the total number of spanning trees of G. We then have the following theorem:

Theorem 3.26. *The function*

$$i(a,b) = \frac{N(s,a,b,t) - N(s,b,a,t)}{N}$$

for all $(a,b) \in E$ defines a unit flow from s to t satisfying Kirchoff's laws.

Remark 3.27. The above expression for $i(a,b)$ is also equal to

$$i(a,b) = \mathbb{P}(T \in \mathcal{N}(s,a,b,t)) - \mathbb{P}(T \in \mathcal{N}(s,b,a,t)).$$

Exactly the same proof as below would work if G is a weighted graph. In this case we would define the weight of a tree to be

$$w(T) = \prod_{e \in T} c(e)$$

and we would set

$$N^* = \sum_{T \in \mathcal{T}} w(T) \quad \text{and} \quad N^*(s,a,b,t) = \sum_{T \in \mathcal{N}(s,a,b,t)} w(T).$$

Then Theorem 3.26 would still be valid with

$$i^*(a,b) = \frac{\mathcal{N}^*(s,a,b,t) - \mathcal{N}^*(s,b,a,t)}{N^*}$$

for all edges (a,b). The negative association theorem would also be true in this case, i.e. when a tree T is picked with probability proportional to its weight.

Proof of Theorem 3.26. It is obvious from definition that i is an antisymmetric function. We next check that it satisfies Kirchoff's node law, i.e. for all $a \notin \{s,t\}$ we have

$$\sum_{x \sim a} i(a,x) = 0.$$

We now count the contribution of each spanning tree T to the sum above. We now consider the unique path from s to t in this spanning tree. If a is a vertex on this path, then there are two edges on the path with endpoint a that contribute to the sum. The edge going into a and the one going out of a. The first one will contribute $-1/N$ and the second one $1/N$. Now if a is not on the path, then there is no contribution to the sum from T. Hence the overall contribution of T is $-1/N + 1/N = 0$ and this proves Kirchoff's node law.

We now check that it satisfies the cycle law. Let $v_1, \ldots, v_n, v_{n+1} = v_1$ constitute a cycle C. We will show that

$$\sum_{i=1}^{n} i(v_i, v_{i+1}) = 0. \tag{3.2}$$

To do this we will work with *bushes* instead of trees. We define an s/t bush to be a forest consisting of exactly two trees T_s and T_t such that $s \in T_s$ and $t \in T_t$. Let $e = (a,b)$ be an edge. We define $\mathcal{B}(s,a,b,t)$ as the set of s/t bushes with $a \in T_s$ and $b \in T_t$.

We now claim that $|\mathcal{B}(s,a,b,t)| = N(s,a,b,t)$. Indeed, for every bush in $\mathcal{B}(s,a,b,t)$ by adding the edge e we obtain a spanning tree of $\mathcal{N}(s,a,b,t)$. Also for every spanning tree $T \in \mathcal{N}(s,a,b,t)$ by removing the edge e we obtain a bush in $\mathcal{B}(s,a,b,t)$.

So instead of counting the contribution of each spanning tree to the sum in (3.2) we look at bushes. Let B be an s/t bush. Then B makes a contribution to $i(a,b)$ of $1/N$ if $B \in \mathcal{B}(s,a,b,t)$, $-1/N$ if $B \in \mathcal{B}(s,b,a,t)$ and 0 otherwise.

So in total an s/t bush B contributes $(F_+ - F_-)/N$, where F_+ is the number of pairs (v_j, v_{j+1}) so that $B \in \mathcal{B}(s, v_j, v_{j+1}, t)$ and similarly for F_-.

But since C is a cycle, if there is a pair (v_j, v_{j+1}) in F_+, then there must be a pair (v_i, v_{i+1}) in F_-. Therefore we get $F_+ = F_-$ and hence the total contribution of B is 0.

Finally we need to check that i is a unit flow, i.e.

$$\sum_{x \sim s} i(s,x) = 1.$$

First we note that $N(s,x,s,t) = 0$ for all $x \sim s$. Every spanning tree must contain a path from s to t, and hence this gives that

$$\sum_{x \sim s} N(s,s,x,t) = N$$

and concludes the proof. □

Proof of Theorem 3.25. We consider G as a network with every edge having conductance 1. Let $e = (s,t)$ be an edge. Then from Theorem 3.26 we get that i is a unit current flow from s to t and

$$i(s,t) = \frac{N(s,s,t,t)}{N},$$

where $N(s,s,t,t)$ is the number of spanning trees that use the edge (s,t). Hence

$$\frac{N(s,s,t,t)}{N} = \mathbb{P}(e \in T).$$

Since the network has unit conductances, we get that

$$i(s,t) = \varphi(s) - \varphi(t),$$

where φ is the potential associated to the unit current i. Therefore the effective resistance between s and t is given by

$$R_{\text{eff}}(s,t) = i(s,t) = \mathbb{P}(e \in T).$$

Let e and g be distinct edges of G. We write $G.g$ for the graph obtained by gluing both endpoints of g to a single vertex. In this way we obtain a one to one correspondence between spanning trees of G containing g and spanning trees of $G.g$. Therefore, $\mathbb{P}(e \in T \mid g \in T)$ is the proportion of spanning trees of $G.g$ containing e. So from the above

$$\mathbb{P}(f \in T \mid g \in T) = R_{\text{eff}}(s,t; G.g).$$

Gluing the two endpoints of g decreases the effective resistance by Rayleigh's principle, and hence

$$R_{\text{eff}}(s,t;G.g) \le R_{\text{eff}}(s,t;G),$$

which is exactly the statement of the theorem. □

Definition 3.28. Let G be a finite connected graph. We write $\mathcal{F}$ for the set of forests of G (subsets of G that do not contain cycles). Let F be a forest picked uniformly at random among all forests in $\mathcal{F}$. We refer to it as USF.

Conjecture 3.29. *For $f, g \in E$ with $f \neq g$ the USF satisfies*

$$\mathbb{P}(f \in F \mid g \in F) \le \mathbb{P}(f \in F)\,.$$

There is a computer aided proof (Grimmett and Winkler) which shows that for graphs on 8 or fewer vertices this conjecture is true.

Theorem 3.30 (Foster's theorem)**.** *Let G be a finite connected network on n vertices with conductances $(c(e))$ on the edges. Then*

$$\sum_{e\in E} c(e) R_{\text{eff}}(e) = n - 1.$$

Proof. Note that if T is a UST in G, then $\sum_{e\in E} \mathbb{P}(e \in T) = n-1$. Using that $\mathbb{P}(e \in T) = c(e)R_{\text{eff}}(e)$ concludes the proof. □

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