Topological solitons: a short course for undergraduates

Prof. Martin Speight
School of Mathematics, University of Leeds, Leeds LS2 9JT
j.m.speight@leeds.ac.uk

21-22/7/25

Rough plan

- Monday
  - Session 1: Lecture "Kinks"
  - Session 2: Lecture "Lagrangian field theory"
  - Session 3: Lecture "Higher dimensions"
  - Session 4: Problems class
- Tuesday
  - Session 5: Lecture "Lumps"
  - Session 6: Lecture "The geodesic approximation"

Exercises for session 4

1. Show that any field theory with Lagrangian density of the form $\mathcal{L}(\phi_t, \phi_x, \phi)$ conserves momentum
$$P = -\int_{-\infty}^{\infty} \frac{\partial \mathcal{L}}{\partial \phi_x} \phi_t dx.$$
Verify that this reproduces the claimed conserved momentum for the sine-Gordon model.

2. Consider the ϕ^4 model:
$$\mathcal{L} = \frac{1}{2}(\phi_t^2 - \phi_x^2) - \frac{1}{2}(1-\phi^2)^2.$$
   (a) Compute its conserved energy E.
   (b) Show that any static field $\phi : \mathbb{R} \to \mathbb{R}$ with kink boundary conditions ($\lim_{x\to\pm\infty} \phi(x) = \pm 1$) has $E \geq \frac{4}{3}$. Construct all static fields that attain this bound. (Hint: repeat the Bogomol'nyi trick.)

3. Consider

$$n(\phi) = \frac{1}{4\pi}\int_{\mathbb{R}^2} \phi \cdot (\phi_x \times \phi_y) dxdy$$

as a functional of the field $\phi : \mathbb{R}^2 \to S^2$. Show that this is a topological invariant of ϕ. That is, show that, for any smooth variation ϕ_s of $\phi = \phi_0$ of compact support,

$$\left.\frac{dn(\phi_s)}{ds}\right|_{s=0} = 0.$$

4. Let $\phi : \mathbb{R}^2 \to S^2$ be a finite energy static solution of the field theory with static energy functional

$$E(\phi) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}(|\phi_x|^2 + |\phi_y|^2) + \frac{1}{4}|\phi_x \times \phi_y|^2 + (1 - \phi_3) \right\} dxdy.$$

Such solutions are called *baby Skyrmions*. Show that

$$\int_{\mathbb{R}^2} |\phi_x|^2 dxdy = \int_{\mathbb{R}^2} |\phi_y|^2 dxdy$$

and

$$\int_{\mathbb{R}^2} \frac{1}{4}|\phi_x \times \phi_y|^2 dxdy = \int_{\mathbb{R}^2} (1 - \phi_3) dxdy.$$

(Hint: The second should come from Derrick's scaling argument. Can you modify this argument to get the first?)

Take home exercises

1. Compute $n(\phi)$, the degree of the map $\phi : \mathbb{R}^2 \to S^2$,

$$\phi(r, \theta) = (\sin f(r) \cos k\theta, sin f(r) \sin k\theta, \cos f(r)),$$

where $f : [0, \infty) \to \mathbb{R}$ is a strictly decreasing function with $f(0) = \pi$ and $\lim_{r\to\infty} f(r) = 0$ and k is an integer. We are using plane polar coordinates here (i.e. $(x, y) = r(\cos\theta, \sin\theta)$).

The obvious strategy is to compute the integral formula for n. This is straightforward if you're familiar with some basic ideas about differential forms, but gets quite laborious if not. A much easier strategy is to count signed preimages. If you're really keen, compute it both ways and check you get the same answer.

2. Assign to each point $(\phi_1, \phi_2, \phi_3) \in S^2$ the stereographic coordinate

$$w = \frac{\phi_1 + i\phi_2}{1 + \phi_3}.$$

   (a) Compute the stereographic coordinates of $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, $(0, 0, -1)$, and mark these coordinates on a diagram of the sphere.

(b) Show that

$$\phi = \frac{(\mathrm{Re}w, \mathrm{Im}w, 1 - |w|^2)}{1 + |w|^2}.$$

(c) Show that the Polyakov equation

$$\phi_x + \phi \times \phi_y = 0$$

rewritten in coordinates $z = x + iy$ on $\mathbb{R}^2$ and w on S^2 is

$$\frac{\partial w}{\partial \bar{z}} = 0.$$

3. Let $\phi : \mathbb{R}^2 \to S^2$ be the general 1-lump solution, that is, in stereographic coordinates

$$w(z) = \frac{a_1}{z + b_1}, \qquad (a_1, b_1) \in (\mathbb{C}^\times \times \mathbb{C}.$$

Show that the energy density

$$\mathcal{E}(x, y) = \frac{1}{2}(|\phi_x|^2 + |\phi_y|^2)$$

of this solution is a lump centred at $z = -b_1$ with width $\sim |a_1|$.

4. Describe $\mathcal{E}$ for the "coincident" 2-lump

$$w(z) = \frac{a_2}{z^2}.$$

Is it just a bigger lump centred at $z = 0$?

5. Let ϕ be a 1-lump, that is

$$w(z) = \frac{q_1 + iq_2}{z + q_3 + iq_4}$$

for some (q_1, q_2, q_3, q_4). Try to compute the metric coefficient g_{11}. What goes wrong?

6. Construct the most general n-lump

$$w(z) = \frac{a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}{z^n + b_1 z^{n-1} + \cdots + b_n}$$

invariant under all rotations

$$w(z) \mapsto \alpha^{-n} w(\alpha z), \qquad \alpha \in \mathbb{C}, |\alpha| = 1.$$

You should find that the space of all such n-lumps forms a submanifold of M_n diffeomorphic to $\mathbb{C}^\times$. Compute the induced metric g on this submanifold. Analyze its geodesic flow.

Further reading

1. N.S. Manton and P.M. Sutcliffe, *Topological Solitons*, Cambridge University Press, 2004. The bible of the subject.

2. Yisong Yang, *Solitons in Field Theory and Nonlinear Analysis*, Springer, 2001. Rigorously develops the underlying mathematical analysis.

3. M. Nakahara, *Geometry, Topology and Physics*, Taylor Francis, 1990. A nice introduction to pretty much all the differential geometry and topology used in theoretical physics.

4. N.S. Manton, "Solitons as elementary particles: a paradigm scrutinized," *Nonlinearity* **21** (2008) T221. A critical appraisal of the idea that elementary particles might really *be* topological solitons.